HARVARD COLLEGE
MATH 22B - FINAL PROJECT

An Introduction to Point-Set Topology

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0 COVER LETTER

Dear Dusty, Math 22b TFs/CAs, and Math 22b students,

We hope this cover letter finds you well! In this paper, we present a short introduction to topology through the lens of set theory, which we covered in the first half of Math 22b. Our goal is to give Math 22b students a primer on how to interpret and prove theorems related to point-set topology by expanding on concepts from class such as open sets, closed sets, and boundary points.

In terms of contributions, we worked together to tackle the proofs while writing other sections independently. Jointly, we wrote up an outline, decided on notation and definitions, talked through proofs, and solved examples. Separately, Nishant drafted the proofs and Erin looked into applications. (Nishant also proudly takes credit for writing the introduction while Erin proudly takes credit for the meme.)

After our first draft, we made several changes according to feedback from peer reviews, including clarifying language used in proofs, correcting typos, and re-structuring the overall paper. We also finished writing the section on homeomorphisms for letters, which involved defining n-vertices, holes, and connectedness, which we then used to classify the alphabet into 8 homeomorphic classes.

We are endlessly grateful to Professor Grundmeier; course TFs and CAs, especially Austin and Kevin for their feedback on our first draft; our peer reviewers Albert, Andy, Kathy, and Guanyu; Albie the Cat; and you, the reader!

Sincerely,
Erin Yuan and Nishant Mishra

1 Introduction and Motivation

It's 8:30 A.M., and you've got class in a half hour. You're a busy mathematics concentrator so... needless to say, you didn't get that much sleep last night. In a desperate attempt to dredge up some level of alertness before your first class, you find your way to Annenberg. As you feebly try to distinguish the coffee mugs and donuts being handed to your fellow first-years, you might be surprised to hear that, for mathematicians, coffee mugs and donuts are technically indistinguishable.

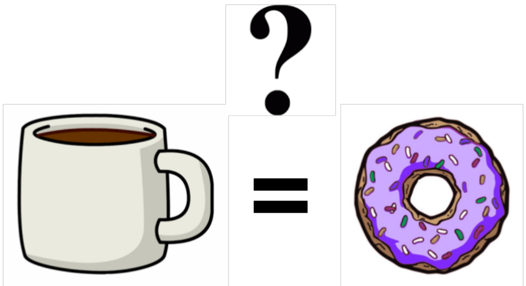

Figure 1.1: A coffee mugs and a donut are considered to topologically be the same. What similarities can you find between the two objects? Figure created by authors.

"Topologists can't tell the difference between a coffee mug and a donut!" This common joke in mathematics draws its humor from the field of **topology**, which studies the properties of a geometric object that are preserved under a series of continuous transformations including stretches, compressions, twists, and other similar manipulations [10]. Building off our knowledge of vector calculus from class, in this paper, we explore how to use these tools to prove and analyze different theorems in **point-set topology**, a sub-field of the broader field of topology. We transition naturally from familiar concepts in Math 22b to investigate the central problem of topology: *what types of objects can be transformed into another?*

1.1 Why Study Topology?

Beyond the core gratification (we hope) that might be gained from proving theorems from the purely theoretical aspects of topology, these concepts have a multitude of applications in subjects beyond mathematics.

One such example is seen in biology. Because the unique functions of proteins draw directly from their folding mechanisms, the interdisciplinary field of **circuit topology** uses mathematical tools to analyze these three-dimensional arrangements. Set theory is used to illustrate the topology of a single folded protein by noting its intramolecular interactions or structural changes during reactions in a matrix [9]. The neat mathematical characterizations of these complex structures are especially useful for biochemical engineers to manipulate molecular structure to create new orientations of molecules—or even new molecules themselves. Topology was also used to understand the "DNA packaging problem": how can a cell's DNA, which measures almost three meters when stretched out, be packed into the tiny nucleus? Since DNA can be visualized as a complicated knot when tightly wound together, biologists use topological concepts from another sub-field called **knot theory**, which studies how loops that intersect can be formed or undone, to see how "knots" in DNA can be wound, untangled, or sliced by enzymes [1].

In data science, mathematics and computer science are married together in a recently developed theory called **topological data analysis**, which describes three-dimensional objects using data based on topological features. Such topological features are defined based on the connectivity of an imaginary ball drawn with radius equal to the distance between any starting point on the surface of the object to any other point on the surface. For example, if the imaginary ball drawn includes a gap between one part of the object and another part of the object, then this feature is deemed a "hole." This information is summarized in a chart (called a "persistence diagram") that computers can read to generate an entire structure's width, breadth, and depth using just numbers [3].

Even beyond academic and scientific disciplines, topology is embedded in our popular culture—particularly the aforementioned discipline of knot theory. As an example, one popular toy called disentanglement puzzles are pieces of string, rings, or wire intertwined together that are meant to be creatively separated [6]. Possibly the most famous type of disentanglement puzzle is the "human knot" game played by large groups during team-building exercises. The goal is to unwind a "knot" created from joining hands randomly with others without breaking any person's grasp. The next time you participate in this activity (COVID-19 restrictions allowing), we hope you'll be reminded of its delightful roots from mathematics!

2 Representing Topology with Set Theory

2.1 What is Point-Set Topology?

Coffee cups don't physically look like donuts, so they can't really be the same, right? However, through the lens of topology, the rules that define whether two objects are the same become much less rigid. To understand these rules, we use **point-set topology**, which lays out the *basic set-theoretic definitions and constructions for topology* [8].

2.2 Spaces in Topology

Recall, from Math 22b, the concepts of **open sets**, **neighborhoods**, **boundary points**, and **closed sets**. As a refresher, here are the definitions of these terms.

Definition 2.1 (Open Set [7])**.** A set $U \subset \mathbb{R}^n$ is called an **open set** if, for every $x_0 \in U$, there exists an $r > 0$ such that $D_r(x_0) \subset U$. Here $D_r(x_0)$ is the open ball centered at x_0 of radius r; namely

$$D_r(x_0) = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$$

.

Definition 2.2 (Neighborhood [7]). A **neighborhood** of a point $x \in \mathbb{R}^n$ is an open set containing x.

Definition 2.3 (Boundary Point [7]). A **boundary point** of a set $U \in \mathbb{R}^n$ is a point x such that every neighborhood of x contains at least one point *in* U and at least one point *not* in U.

Definition 2.4 (Closed Set). A set U is called a **closed set** if it contains all of its boundary points.

Something that repeatedly comes up in the definitions of open and closed sets—specifically regarding the positive radii of balls constructed around a point in the set—is the notion of *distance*. We can formalize this notion of distance by characterizing the space where open and closed sets typically exist: a **metric space**.

Definition 2.5 (Metric Space [4]). A **metric space** is a set X that is made of a) a set of *points* and b) a function called the *distance function or metric*, which takes the distance between any pair of points in the set. For example, the distance between two points α and β in X is notated as $d(\alpha,\beta)$, where d is the distance function. The distance function, possesses the following three characteristics:

1. (Positive Definite). $d(\alpha,\beta) \geq 0$ for any α and β in X, and $d(\alpha,\beta) = 0$ if and only if α and β are equal and are in X.

2. (Symmetry). $d(\alpha,\beta) = d(\beta,\alpha)$ for any α and β in X.

3. (The Triangle Inequality). $d(\alpha,\beta) + d(\beta,\gamma) \geq d(\alpha,\gamma)$ for any α, β, and γ in X.

Our definitions from Math 22b of open sets, closed sets, and boundary points only applies *in the context of metric spaces* because metric spaces include the crucial concept of distance. However, topology is intrinsically much more abstract, and the spaces in topology do not necessarily operate with distance functions like metric spaces. We now generalize the idea behind metric spaces to new spaces called **topological spaces**, first by introducing the definition of a **topology**.

Definition 2.6 (Topology [4]). A **topology** τ on a set X consists of subsets of X satisfying the following three properties:

1. Both the empty set and X are elements of τ.

2. Any union of elements of τ is an element of τ.

3. Any intersection of finitely many elements of τ is an element of τ.

Definition 2.7 (Topological Space [4]). A **topological space** is a *pair* (X,τ), where X is a set and τ is a topology.

Let's apply our new knowledge of topologies and topological spaces to an example.

Example 1. Let $X = \{1,2,3,4\}$ and $\tau = \{\emptyset,\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$. Is τ a valid topology?

Solution. We check the three properties of a topology listed above:

1. *Both the empty set and X are elements of τ.* Both $\emptyset$ and $X = \{1,2,3,4\}$ are indeed included in τ.

2. *Any union of elements of τ is an element of τ.* In the list $\{\emptyset\}, \{1\}, \{1,2\}, \{1,2,3\}$, and $\{1,2,3,4\}$, each element is a *subset* of the element to its *right* {i.e. $\{1\}$ is an element of $\{1,2\}$, $\{1,2\}$ is an element of $\{1,2,3\}$, etc.}. Thus any union of any number of elements of τ will equal the element in the union that is *furthest along* in the list {in other words, the "right-most" element). Since each element composing the union has to be from τ, then any union of elements in τ is an element of τ.

3. *Any intersection of finitely many elements of τ is an element of τ.* In the list $\{\emptyset\}, \{1\}, \{1,2\}, \{1,2,3\}$, and $\{1,2,3,4\}$, each element is a *superset* of the element to its *left.* Thus any intersection of any number of elements of τ will equal the element in the intersection that is *earliest* in the list (or the "left-most" element). Since each element composing the intersection has to be from τ, then any intersection of elements in τ is an element of τ.

So τ is a valid topology of X! This conclusion also lets us say that (X,τ) is a topological space.

Note that while the notion of distance doesn't appear in the definitions for topologies or topological spaces, that *doesn't* necessarily mean that distance *doesn't* exist in topological spaces—only that we can't *assume* that it does. We now want to expand our previous definitions of Math 22b concepts to function within these new spaces. However, since we originally defined these terms using distance (with open balls and radii), and we now can't assume that distance exists, we will redefine these concepts *without distance* so they fit in the context of point-set topology.

Definition 2.8 (Open Sets, *Topological Form* [4])**.** A set is considered an **open set** in (X,τ) if it is in τ.

We can remove the metric assumption from neighborhoods by just removing $\mathbb{R}^n$ from the original definition and generalizing it to X.

Definition 2.9 (Neighborhood, *Topological Form* [4])**.** A **neighborhood** of a point $x \in X$ is an open set in X containing x.

We also follow the same idea (generalizing to X) to arrive at a new definition for boundary points.

Definition 2.10 (Boundary Point, *Topological Form* [2])**.** A **boundary point** of a set $A \in X$ is a point x such that every neighborhood of x contains at least one point in A and at least one point not in A.

For closed sets, let's first say the following:

Definition 2.11 (Closed Sets, *Topological Form* [4])**.** A set A that is contained in another set X is considered a **closed set** if $X \setminus A$ is open.

We've now refined our original definitions to ensure that they do not rely on distance. This process of reworking the definitions for neighborhoods and boundary points might make intuitive sense—after all, we simply generalized $\mathbb{R}^n$ to X to make sure there was no metric assumption. But what about open and closed sets? How can we be sure that our new definitions capture the same meaning as our old ones? We can't just paste in new, random definitions for established terms–we need to make sure that these new definitions imply the same things as the old ones! If we can show that our new definition implies the conditions in metric spaces, *we can link our two definitions of closed sets*, where the topological variants of the definitions are just a *more abstract version* of the metric variants. We now prove this exact claim.

Theorem 2.1. *If X is a topological set and A is a closed set in X (as detailed in Definition 2.11), then A contains all of its boundary points. [4]*

Proof. Let A be a closed set in the manner that it is described in Definition 2.11 (it contains all of its boundary points). We need to show that all the boundary points of A are in A. If A has no boundary points, the claim is vacuously true. But what if A *does* have boundary points? Let α be a boundary point of A. We proceed using proof by contradiction.

By way of contradiction, let us assume that α is not in A. Since A is closed, then$X$ is open, as defined in Definition 2.8 (the set is in τ, the topology of X). So α is in an open set that is in τ. This, however, is a contradiction, since this implies that α has a neighborhood that does not contain points in A, even though α is a boundary point, meaning that any open set with α in it *needed* at least one point in A. Therefore, our initial assumption is false: in the context of topological spaces, if A is closed, it contains all of its boundary points. ■

Delightfully, our two definitions for closed sets corroborate each other! Since we used our proposed topological definitions of both open and closed sets to indicate the same conclusions that our metric-space closed set definitions do, we can conclude that our open set definitions are consistent with each other as well. With this powerful connection, we can view metric spaces as a *special case* of topological spaces.

2.3 Continuity in Topology

Since distance can't be assumed in topological spaces, we also must redefine **continuity**, which uses the $\delta - \epsilon$ scheme in traditional vector calculus. Believe it or not, we've not only already seen the topological definition, but we've also proved that it's equivalent to the calculus version! (See Problem Set 2, Proofy Problem 2.) As a refresher, here is the definition again:

Definition 2.12 (Continuity, *Topological Form* [4])**.** A function $f : X \to Y$ is **continuous** if and only if, for all U open in Y, the pre-image of U is open in X.

3 Homeomorphisms

3.1 Definition and Theorems

With the new definition of continuity, we can use it to uncover a topological form for the concept of **equality**, which is less strict than the form we're used to. The relations of equality in topology are called **homeomorphisms**, which we will now formally define.

Definition 3.1 (Homeomorphism [4])**.** A **homeomorphism** is a function $f: X \to Y$ between two topological spaces X and Y that

1. is a continuous bijection, and
2. has a continuous inverse function f^{-1}.

Homeomorphisms are the topological equivalent of the concept of equality—however, do the properties of equality translate to define the properties of homeomorphisms? Let's start by trying to prove that compositions of homeomorphisms are homeomorphic.

Theorem 3.1. *If $f: X \to Y$ is a homeomorphism and $g: Y \to Z$ is another homeomorphism, then the composition $g \circ f: X \to Z$ is also a homeomorphism. [4]*

Proof. Let f and g be homeomorphisms. We need to prove $g \circ f: X \to Z$ is a homeomorphism, so we want to show that the two conditions of a homeomorphism hold; namely:

1. $g \circ f: X \to Z$ is continuous
2. There exists an inverse of $(g \circ f)$ that maps $Z \to X$ and it is continuous.

Condition 1. Since f is a homeomorphism and g is a homeomorphism, both are continuous functions. Hence $g \circ f$ is also continuous, since we proved in 22a that the composition of continuous functions are also continuous.

Condition 2. Again, since f is a homeomorphism and g is a homeomorphism, their respective inverses, f^{-1} and g^{-1}, exist and are continuous. So $f^{-1} \circ g^{-1}$ must also exist. We verify:

$$\begin{aligned} &(g \circ f) \circ (f^{-1} \circ g^{-1}) \\ &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= g \circ g^{-1} \\ &= I_Y. \end{aligned}$$

Similarly:

$$\begin{aligned} &(f^{-1} \circ g^{-1}) \circ (g \circ f) \\ &= f^{-1} \circ (g^{-1} \circ g) \circ f \\ &= f^{-1} \circ f \end{aligned}$$

$$= I_X.$$

So our proposed function, when given the original function has an input, results in an identity matrix. The same results when the original function is given the proposed function as an input. So there exists a continuous inverse of $(g \circ f)$ that maps $Z \to X$. Since both conditions are met, then we can conclude that $g \circ f$ is a homeomorphism. ∎

In the proof above, we used homeomorphisms to classify *functions*. We can also use homeomorphisms to classify *spaces*: the term **homeomorphic** is used for spaces if a homeomorphism exists between them. We formalize this in the following definition.

Definition 3.2 (Homeomorphic [4])**.** Two topological spaces X and Y are **homeomorphic** if there exists a *continuous* map $f : X \to Y$ and a continuous inverse $f^{-1} : Y \to X$, implying that $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$.

Now that we can classify spaces as homeomorphic, perhaps you may wonder: are the properties of "equality" of homeomorphisms enough to obey the rules of **equivalence relations**, a concept from Math 22a? We introduce and prove the following theorem on homeomorphisms and equivalence relations.

Theorem 3.2. *A homeomorphism forms an equivalence relation on the class of all topological spaces [4]. More specifically, homeomorphisms meet the three criteria for an equivalence relation:*

1. *(Reflexivity). X is homeomorphic to X.*
2. *(Symmetry). If X is homeomorphic to Y, then Y is homeomorphic to X.*
3. *(Transitivity). If X is homeomorphic to Y, and Y is homeomorphic to Z, then X is homeomorphic to Z.*

Proof. We prove that each of the conditions hold.

Condition 1: Reflexivity. To show that X is homeomorphic to X, we need to determine if there exists a *continuous* map $f : X \to X$ and its inverse, $f^{-1} : X \to X$, implying that $f \circ f^{-1} = I_X$ and $f^{-1} \circ f = I_X$. Since the input and output spaces are the same, we just take the identity map as our function f.

Condition 2: Symmetry. Let X be homeomorphic to Y. This implies that there exists some function $f : X \to Y$ between X and Y such that f is a continuous bijection and has a continuous inverse function f^{-1}. To show that Y is also homeomorphic to X, we want to find a function $g : Y \to X$ between Y and X such that g is a continuous bijection and has a continuous inverse function g^{-1}. Let's set the inverse function f^{-1} as the function $g : Y \to X$, and the function f as the inverse function $g^{-1} : X \to Y$. We know that $g = f^{-1}$ is a continuous bijection since f^{-1} is continuous and is a bijection because f is a bijection. Also, $g^{-1} = f$ is continuous since f is continuous. So there must exist some function $g : Y \to X$ between Y and X such that it is a continuous bijection and has a continuous inverse function g^{-1}. Thus Y is homeomorphic to X.

Condition 3: Transitivity. Recall from Theorem 3.1 that if $f : X \to Y$ is a homeomorphism and $g : Y \to Z$ is another homeomorphism, then the composition $g \circ f : X \to Z$ is also a homeomorphism. Since we can take $g \circ f : X \to Z$ as a homeomorphism from X to Z, then X and Z are homeomorphic.

Thus homeomorphisms satisfy all three conditions for equivalence relations on the class of all topological spaces. ■

The classes of all topological spaces are also called **homeomorphic classes** [4]. In our example section, we will explore the physical implications of two objects being in the same homeomorphic class.

3.2 Examples of Homeomorphisms

To peel away the abstractness of homeomorphisms, it's important to show how they can be applied to some concrete functions. Let's launch into an example.

Example 1. Let $X = (1, \infty)$ and $Y = (0, 1)$. Are these spaces homeomorphic?

Solution. Recall that two topological spaces are considered homeomorphic if there exists a continuous function $f : X \to Y$ and its inverse $f^{-1} : Y \to X$. First, we must identify a function f that maps from X to Y. Since we want this function to take inputs from 1 to ∞, but remain bounded between 0 and 1, let's look into an *asymptotic function.* We try the simplest one: $f(x) = \frac{1}{x}$. Indeed, this function takes any value from $(1, \infty)$ as an input and spits out a value $(0, 1)$, which makes it a candidate for our homeomorphisms! We now verify this statement.

By Definition 3.1, to show that $f : X \to Y$ is a homeomorphism, the following conditions must be met:

1. f is a continuous bijection, and
2. f has a continuous inverse function f^{-1}.

Condition 1. During lecture, we showed that f is continuous. To show that f is bijective, we prove that it is both injective and surjective.

Injective. Let x, y be values in X such that $f(y) = f(x)$. Then $\frac{1}{y} = \frac{1}{x}$, so $x = y$. Thus f is injective.

Surjective. Let y be an element in Y. We can easily derive an x in X such that $f(x) = \frac{1}{x}$ by setting $x = 1/y$ so $f(y) = \frac{1}{\frac{1}{y}} = y$. Since y can't be 0, there's no danger of deriving an undefined value. Thus f is surjective.

Condition 2. f^{-1} can be represented as $g(x) = \frac{1}{x}$. Again, we know from our work in 22b that $\frac{1}{x}$ is continuous, so f has a continuous inverse function.

Therefore f is a valid homeomorphism, so X and Y are homeomorphic. This example may seem simple, but it's actually quite illuminating considering that X was bounded but Y was not. In topology, then, we can essentially **stretch** a bounded set to match up with an unbounded one!

We can apply this idea of "stretching" to the coffee mugs and donut example from before. The graphic below visually shows how a topologist could see a coffee mug and donut as one and the same!

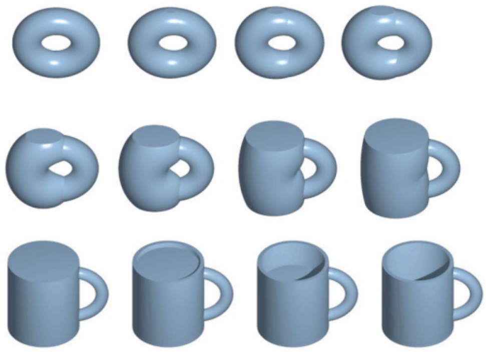

Figure 3.1: The donut (top left) slowly transforms to a coffee mug (bottom right) [10].

4 Homeomorphisms Between Letters of Alphabet

As a culminating exercise, we apply our study of homeomorphisms to categorize the twenty-six letters of the English alphabet used to write this very paper. (To standardize our analysis due to the many typefaces that exist, and to follow convention, we use the Sans Serif LaTeX font to represent letters in this section [4].)

4.1 Holes, Vertices, and Connectedness

To build the background necessary to complete this exercise, however, let's first explore how we can use the idea of "stretching" to classify letters. Notice that, from the donut-to-coffee-mug example, there is one core similarity between the two that remained consistent even while being transformed: *the hole!* The reason is because these transformations don't allow us to tear apart the surfaces in question, so we can never get rid of the hole via the deformation used to stretch the donut into a coffee cup. We generalize this observation in the following definition.

Definition 4.1 (Holes [5])**.** A **hole** in a mathematical object is a topological structure that prevents the object from being continuously shrunken to a point.

We now have one method of classifying letters—by their number of holes—since no topological deformations can remove these holes. For example, P has one hole at its top while B has two holes stacked on top of each other. Thus P and B must belong to separate "groups" in our classification.

But are there other methods to classify letters on an even more specific level? In other words, are there any distinguishing features in letters besides the presence of holes? The answer is yes.

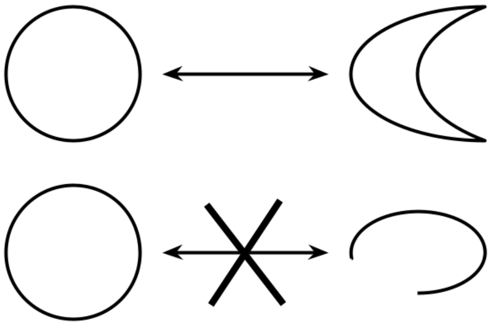

Figure 4.1: An object with one hole is homeomorphic to any other object with one hole (top row), and an object with one hole cannot be homeomorphic to an object with no hole (bottom row). Figure by authors adapted from [4].

In the letter P, besides having one hole, it also has another distinguishing feature: *the number of intersections between different curves* in the letter.

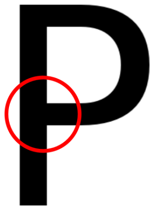

Figure 4.2: The one intersection in the letter P is highlighted by the red circle. Figure created by the authors.

Even though such intersections—also known as **vertices**—can be visually identified, we'd like to formalize what they are in topology (just as we did with holes). To do that, let's first delve into the idea of spaces being **connected**.

Intuitively, we say a space is connected if it *exists in one piece*, meaning that we *cannot split* the space into the union of two **disjoint** open sets and their respective boundary points. A more rigorous definition follows:

Definition 4.2 (Connected [4])**.** A space X is **connected** if, when decomposed as the union $A \cup B$ of two nonempty subsets, then $(A \cup A^{\beta}) \cap B \neq \emptyset$ and $A \cap (B \cup B^{\beta}) \neq \emptyset$, where A^{β} are the boundary points of A and B^{β} are the boundary points of B.

Before using connectedness to define intersections in our letters, we first need to prove that homeomorphisms preserve connectedness. That way, if we use the notion of connectedness to define vertices, we can argue that the intersections in a surface cannot be eliminated via continuous deformation.

Theorem 4.1. *If $f : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected [4].*

In order to prove the theorem, we first prove a couple of lemmas.

Lemma 4.2. If X is connected, then the only subsets of X that are both open and closed are X and $\emptyset$ [4].

Proof. Let X be connected. Assume, by way of contradiction, that there exists a set $A \neq \emptyset$ and $A \neq X$ such that it is *both* an open and closed subset of X. Let B be a set such that $B = X \setminus A$. This implies that $A \cap B = \emptyset$. Because A is closed, B is open, and since A is open, B is closed. So B is *both* open and closed. Since both A and B are closed, both sets contain their respective boundary points:

$$A = A \cup A^{\beta}, B = B \cup B^{\beta}.$$

It then follows that

$$\emptyset = A \cap B = (A \cup A^{\beta}) \cap B = A \cap (B \cup B^{\beta}).$$

This is a contradiction since we assumed X is connected. Thus the initial assumption must be false, so the only subsets of X that are both open and closed are X and $\emptyset$. ■

We use the previous lemma to prove the next lemma:

Lemma 4.3. The continuous image of a connected space X is connected [4].

Proof. Let $f : X \to Y$ be a surjective continuous function. If $A \subset Y$ such that A is both open and closed, then $f^{-1}(A)$ is also both open and closed since f is continuous. Since $f^{-1}(A)$ is both open and closed and X is connected, then $f^{-1}(A)$ must either be equal to X or $\emptyset$, as per Lemma 4.2. Therefore A is either equal to Y or $\emptyset$, so Y is connected. ■

We now return to our proof of the theorem, which follows naturally from the lemmas above.

Proof. Since the continuous image of a connected space X is connected, we can immediately conclude that if $f : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected. ■

We apply our understanding of connectedness to formalize the idea of intersections, or vertices.

Definition 4.3 (n-vertex [4])**.** An n**-vertex** in a subset L of a topological space S is an element $v \in L$ such that there exists some neighborhood $N_0 \subset S$ of v where all the neighborhoods $N \subset N_0$ of v satisfy the following properties:

1. $N \cap L$ is connected.
2. The set formed by removing v from $N \cap L$ is *not* connected, and is composed of exactly n disjoint sets, each of is connected.

Notice that the floor is open for multiple types of vertices! There can be 4-vertices, 3-vertices, and even 2-vertices. (While the 2-vertex does exist, the idea is somewhat useless to us in the context of classifying letters of the alphabet. We could simply pick any point in the curve that isn't intersecting another curve and call it a 2-vertex, so there would be an infinite number for each letter. Thus when classifying surfaces, we will stick to an n of at least 3 for n-vertices.)

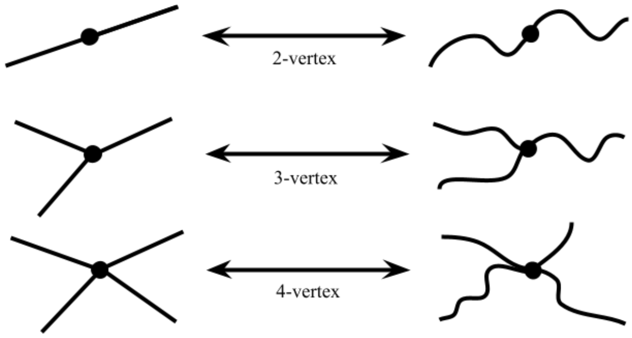

Figure 4.3: Examples of a 2-vertex (top), 3-vertex (middle), and 4-vertex (bottom) preserved via homeomorphisms. Figure by authors adapted from [4].

4.2 Classifying the Alphabet

Since we proved earlier that homeomorphisms preserve connectedness, we know that, just like with holes, n-vertices depend entirely on the topology of a surface. What follows is a powerful conclusion. If two letters in our alphabet have the *same number of n-vertices and holes*, we can categorize them into the *same homeomorphic class*. Within a single class, we can also *continuously deform any letter in the class* to become *any other letter in the same class*. We are now ready to classify the letters of the English alphabet into 8 resulting classes:

1. (0 holes, 0 3-vertices, 0 4-vertices): C G I J L M N S U V W Z
2. (1 hole, 0 3-vertices, 0 4-vertices): D O
3. (0 holes, 1 3-vertex, 0 4-vertices): E F T Y
4. (1 hole, 1 3-vertex, 0 4-vertices): P
5. (0 holes, 2 3-vertices, 0 4-vertices): H K
6. (1 hole, 2 3-vertices, 0 4-vertices): A R
7. (2 holes, 2 3-vertices, 0 4-vertices): B
8. (0 holes, 0 3-vertices, 1 4-vertex): X
9. (1 hole, 0 3-vertices, 1 4-vertex): Q

We invite the reader to explore homeomorphisms between letters of their own unique handwriting, or between the alphabets or characters of other languages.

5 Conclusion

Our examination of point-set topology gives us a toolkit to further explore the broader field of topology. Using the presented definitions of metric spaces, topological spaces, and homeomorphisms from extending set theory to point-set topology, we have explored the connection from Math 22b concepts in a novel context.

To delve further into this field, there are several natural extensions of point-set topology into other topology disciplines. For example, **algebraic topology** extends upon homeomorphic-like classifications to define other topological invariants like homotopies, homologies, and cohomologies that can also be used to categorize alphabet letters. **Differential topology** specifically studies objects with "smooth" structures, while geometric topology focuses on a type of shape called a manifold. Along with the applications presented in the introduction to biology, computer science, and even our everyday lives, the mathematics of topology yield a rich and versatile tool that we hope may be part of your future.

Figure 5.1: Figure courtesy of r/mathmemes.

References

[1] M. Beals, L. Gross, and S. Harell. *DNA and Knot Theory*. URL: http://www.tiem.utk.edu/~gross/bioed/webmodules/DNAknot.html.

[2] *Boundary Point*. URL: https://mathworld.wolfram.com/BoundaryPoint.html.

[3] Mathieu Carrière. *A concrete application of Topological Data Analysis*. May 2019. URL: https://towardsdatascience.com/a-concrete-application-of-topological-data-analysis-86b89aa27586.

[4] Renzo Cavalieri. *Introduction to Topology*. 2007. URL: https://www.math.colostate.edu/~renzo/teaching/Topology10/Notes.pdf.

[5] *Hole*. URL: https://mathworld.wolfram.com/Hole.html.

[6] Matthew Horak. "Disentangling Topological Puzzles by Using Knot Theory". In: *Mathematics Magazine* 79.5 (2006), p. 368. DOI: 10.2307/27642974.

[7] Jerrold E. Marsden, Karen Pao, and Anthony J. Tromba. *Vector calculus*. Freeman, 1996.

[8] *Point-Set Topology*. URL: https://mathworld.wolfram.com/Point-SetTopology.html.

[9] Otto Schullian et al. "A Circuit Topology Approach to Categorizing Changes in Biomolecular Structure". In: *Frontiers in Physics* 8 (2020). DOI: 10.3389/fphy.2020.00005.

[10] *Topology*. Apr. 2021. URL: https://en.wikipedia.org/wiki/Topology.