HARVARD COLLEGE
MATH 22A - FINAL PROJECT

An Introduction to the Applications of Linear Algebra in Graph Theory

Nishant Mishra, Erin Yuan
Fall 2020

0 COVER LETTER

Dear Dusty, Math 22A TFs/CAs, and Math 22A readers,

We hope this cover letter finds you well! In this paper, we present a short introduction to graph theory through the lens of linear algebra. Specifically, our goal is to give our Math 22A audience a primer on how to interpret graphs in more abstract terms using only linear algebra by proving theorems involving eigenvalues, matrices, and other concepts.

In terms of contributions, we worked together to tackle the proofs while writing other sections independently. Jointly, we wrote up an introduction, decided on notation, talked through proofs, and solved examples. Separately, Nishant researched the MST application and drafted the proofs; Erin looked into other applications and created graphics.

After we received our peer review feedback, we wrote up a clearer guiding narrative (*Charles M.*); expounded upon concepts like the spectral theorem, simple graphs, and matrices indices, as well as added another illuminating example (*Iris C.*); fixed image citations (*Lillian P.*); reworked the ordering of sections (*Parita S.*); adopted the typographical conventions for definitions/remarks (*Jamin L.*); and fixed the typos our reviewers collectively uncovered.

Finally, we'd like to point out that one linear algebra concept we use minimally, trace, might be unfamiliar to a Math 22A student. However, we are careful to construct a clear definition of trace before applying it, and we hope our explanations add sufficient background to understand the proofs that involve trace.

We are endlessly grateful to Prof. Grundmeier; the course TFs and CAs; our draft reviewers Charles, Iris, Lillian, Parita, and Jamin; Albie the Cat; and you, the reader!

Sincerely,
Nishant Mishra and Erin Yuan

1 Introduction and Motivation

Say you are an early 18th century traveler to Königsberg, a Prussian port city by the Baltic sea. You hope to spend the day exploring the city, which is made of two landmasses connected by a series of seven bridges. To do as much sightseeing as possible, you plan on crossing each of the seven bridges, but with a catch: to save time, each bridge can only be crossed once and no more than once. Is it possible?

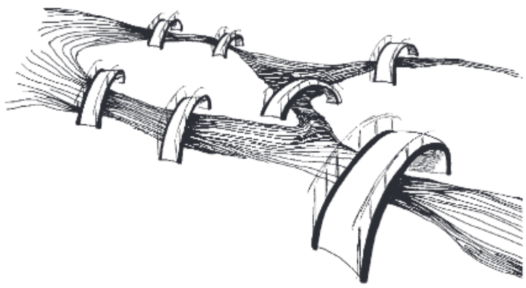


Figure 1.1: The Seven Bridges of Königsberg [4]. Can you devise a way to cross all seven bridges only once?

This problem, commonly referred to as **The Seven Bridges of Königsberg**, was eventually proven rigorously by the mathematician Leonhard Euler in 1736 to have no solution [3]. His reasoning laid the foundations for modern graph theory—the mathematical study of graphs, or pairwise relations between objects—which has since branched out extensive applications in a variety of contexts. Among the more powerful tools that mathematicians have used to study this field is linear algebra, the branch of mathematics that seeks to examine linear sets of equations.

While it may initially seem difficult to reconcile the concrete concept of a graph with the abstract nature of linear algebra, studying graphs under this lens enables us to derive some surprising results. To present an introduction to graph theory, we apply key ideas from linear algebra, including matrix operations, eigenvalues, and the Spectral Theorem, to expand upon our previous notions of the mathematical definition of a "graph," which has real-world uses in disciplines ranging even beyond science.

1.1 What Exactly is a Graph?

Up to this point, we have viewed graphs in the familiar geometric R_n space. For example, within the Cartesian xy-plane, it is natural to think of a graph as a plot of a function, potentially falling under the umbrella of families like the sinusoidal, polynomial, or Limaçon curves. However, because this previous view is quite restrictive, we now open up the definition of a graph to the more abstract terms of graph theory.

Definition 1.1 (Graph, Vertex, and Edge)**.** A **graph** is a structure containing elements in which *some pairs of elements are related in a certain way.* We refer to these *elements* as **vertices**, and *any possible relations that might exist between vertices* as **edges** [8].

Visually, vertices are represented by circles and edges are represented by lines connecting any two circles. Note that graphs can be **infinite** or **finite** according to its number of vertices or edges, although for our purposes we limit ourselves to only finite graphs with finite vertices and edges.

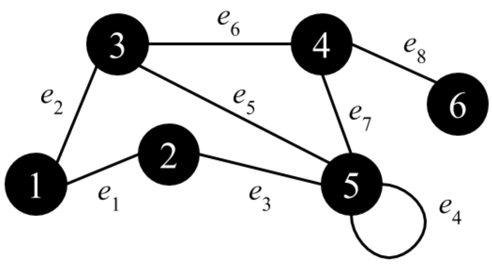


Figure 1.2: A pictorial representation of a graph. Each of the six circles is a vertex. The eight lines labeled $e_1, \ldots, e_8$ are edges. Note that not every pair of vertices has to have an edge in between them (for example, there is no edge between circles 2 and 3). Also, an edge can circle back to the same vertex (see e_4). Figure by authors.

1.2 Why Study Graphs?

In addition to the core gratification (we hope) one might obtain from proving theorems of the purely theoretical aspects of graph theory, such concepts have a multitude of applications in subjects beyond mathematics. One excellent example is the **Minimum Spanning Tree (MST)**, which, put simply, is a graph that seeks to connect each of its vertexes in a way that minimizes the **edge weight**. The meaning of "edge weight" depends on the scenario—if we utilize an MST to plot out different houses (the vertices) in a set of roads (the edges), one could say that the longer the road distance is between two houses, the larger the edge weight is. By plotting out paths that minimize this edge weight, we can create useful algorithms that calculate the MST for minimizing cost, whether in the case of planning the cheapest route to deliver multiple packages, laying down telecommunications cable in a neighborhood, or planning functions of ride-sharing services [5].

Graphs have also proven their benefits in medically-oriented fields. In chemistry, graphs are commonly used to generate molecular models, with vertices acting as atoms and edges as bonds. Since many physical properties, like boiling point, are directly related to structure, such graphical representations allow ease of calculations not only for larger collections of known molecules but also for molecules that may be hard to create in a traditional laboratory [2]. The significance of graphs in biology hits close to home due to the current challenges facing our world: with vertices for regions and edges for migration paths, these graphs can trace disease propagation to predict infectious spread. In molecular and cellular biology, graphs are used to create "bio-networks" that tabulate protein interactions or transcription of genes in live cells [7].

Even beyond traditional scientific disciplines, graphs serve to be surprisingly versatile in portraying relationships. As an example, sociologists construct "social networks" with people as vertices and degrees of familiarity as edges to study influence, friendship, or communication. Altogether, the flexibility of graph theory to adapt to multiple subjects demonstrates its importance as an object of study.

2 Representing Graphs with Linear Algebra

2.1 Graphs as Sets/Functions

In order to begin studying graph theory in a linear algebra context, we first translate the core features of a graph into a more mathematical form. A finite graph G becomes a collection of three attributes: (1) vertices, (2) edges, and (3) the placement of those edges between the vertices.

Definition 2.1 (Vertex Set)**.** The **vertex set** is the set of n vertices of a graph that is represented as $V = \{v_1, ..., v_n\}$. As a shortcut, a pair of vertices, say $\{v_i, v_j\}$, is often shortened to simply $\{i, j\}$).

Definition 2.2 (Edge Set)**.** The **edge set** is the set of m edges of a graph that is represented as $E = \{e_1, ...e_m\}$.

Definition 2.3 (Incidence Function)**.** The **incidence function** of a graph is a function $\phi : E \to (\binom{V}{2})$ that takes an element from the edge set E and maps it to an element of $(\binom{V}{2})$, which represents all possible pairs of vertices in V, even a pair consisting of the same vertex twice! [8]

Because the notation of $(\binom{v}{2})$ from Definition 2.3 may be confusing, we demonstrate this idea with the following aside.

Aside 2.1. Suppose we have a set $S = \{1,2,3\}$. Then $\binom{S}{2} = \{\{1,2\},\{2,3\},\{1,3\}\}$, as per the rules of basic combinatorics. But we want this set to also have pairs consisting of the same vertex twice, so we attach an extra parenthesis to $\binom{S}{2}$ as notation representing this expanded set in question. So $(\binom{S}{2}) = \{\{1,1\},\{2,2\},\{3,3\},\{1,2\},\{2,3\},\{1,3\}\}$. If we define the function $\phi : E \to (\binom{V}{2})$, then each element of the edge set maps to the pair of vertices in $(\binom{V}{2})$ that the edge connects. Looking back at our example graph from Figure 1.1, we would say $\phi(e_1) = \{1,2\}$, and $\phi(e_4) = \{5,5\}$.

Putting these definitions together, we can describe a finite graph as $G(V,E,\phi)$. We now expand upon some other relevant terminology.

Definition 2.4 (Adjacent)**. Adjacent** vertices are vertices with *at least one edge connecting them.*

Definition 2.5 (Incident)**.** An **incident** edge between two vertices is *an edge that connects two adjacent vertices.*

Definition 2.6 (Multiple Edge)**.** If two adjacent vertices have *more than one edge incident to them*, we say that there is a **multiple edge** between them.

Definition 2.7 (Loop)**.** An *edge that connects a vertex to itself* is referred to as a **loop**.

Definition 2.8 (Simple Graph)**.** A graph G is considered to be **simple** if it has *no loops or multiple edges.*

Definition 2.9 (Complete Graph)**.** A graph is considered to be **complete** if there exists exactly one edge between any two distinct vertices. Complete graphs can be uniquely defined by their number of vertices. Here, let K_p represent the complete graph of p vertices.

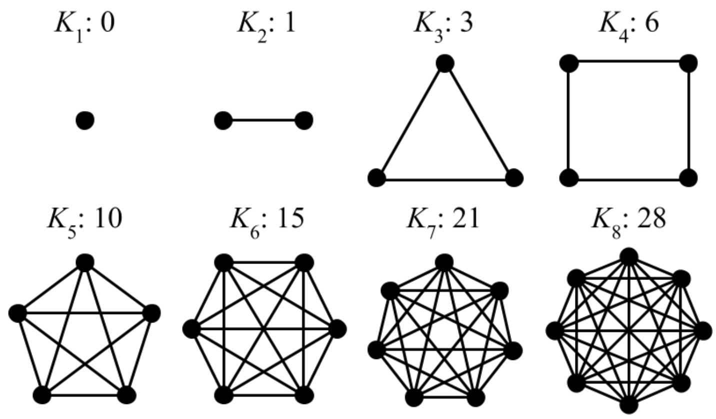
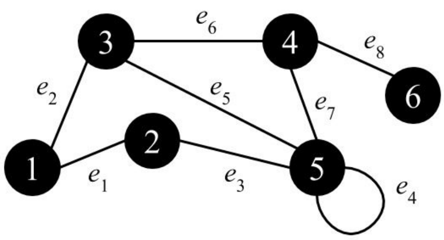


Figure 2.1: Examples of complete graphs of $1, \ldots, 8$ vertices. The number of edges is written after each labeled K_p and can be found using $\binom{p}{2} = \frac{p(p-1)}{2}$ since there are no multiple edges in a complete graph and every distinct pair of vertices will have an edge between them. Figure by authors.

2.2 The Adjacency Matrix

While we have translated graphs into set theory, we have yet to put them into a form that represents the heart of linear algebra—the matrix. How can we do this? If we take a look at a simple graph, it is fairly easy to count and identify which vertices are adjacent to each other. Since each edge has two vertices, and a matrix has rows and columns, we can represent a graph under a matrix scheme.

Definition 2.10 (Adjacency Matrix)**.** An **adjacency matrix** A for a finite graph with n vertices is an n by n matrix where the entry in the ith row and the jth column is the *number of edges* that connect the ith and jth vertices. The name "adjacency" originates from the fact that the matrix tracks the number of edges between adjacent vertices [8].



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2.2: A reexamination of the graph from Figure 1.1, this time linking its visual representation with its adjacency matrix. Notice that there is a loop in the graph from the edge labeled e_4 that connects back to circle 5, which shows up as a 1 on the diagonal at the entry in the 5th row and 5th column.

We note several interesting observations about the adjacency matrix that are summarized in the following remarks.

Remark 2.1. The way we number the vertices of a graph is arbitrary. Given a graph with n vertices, we can number them from 1 to n in any order. So a graph can have *more than one* valid adjacency matrix.

Remark 2.2. The number of edges connecting the ith and jth vertices is equal to the number of edges connecting the jth and ith vertices. So any adjacency matrix A must be *symmetric* across the main diagonal since we assume that all edges are bidirectional (we can traverse through them both ways). Thus the adjacency matrix is equal to its transpose: $A = A^T$.

Remark 2.3. Since we know that each entry on the main diagonal must have the same row and column index, we can say that *any nonzero element on the main diagonal represents a loop* connecting a vertex to itself.

Remark 2.4. We can also use the adjacency matrix to classify a simple graph. Recall that a simple graph can at most have only one edge between vertices. So any entry in the adjacency matrix of a simple graph must have the *maximum value of 1.* Additionally, since simple graphs do not have loops, *all entries on the main diagonal must be 0.*

3 n-STEP WALKS

3.1 WHAT IS AN n-STEP WALK?

Armed with these powerful tools to represent graphs mathematically, we can now begin applying what we know to a case study—specifically, the n-step walk.

Definition 3.1 (n-step walk)**.** An **n-step walk** from vertex v_x to v_y is *a sequence that uses n edges and the vertices between them to take us from v_x to v_y*. More rigorously, this sequence can be denoted as $\{v_1, e_1, v_2, ...e_n, v_{n+1}\}$ where $v_1 = v_x$ and $v_{n+1} = v_y$ and each of edges in the sequence is the edge connecting the vertex before and after it [8].

There can be many paths from one vertex to another, so there can be multiple valid n-step walks between any two pairs of vertices. One could use trial-and-error to compute the total number of n-step walks between two vertices, but this method quickly becomes increasingly infeasible when tested with larger graphs and/or longer n-step walks. The adjacency matrix, as we will soon find, provides us with a more systematic way to compute the number of n-step walks between any two vertices.

Theorem 3.1. *For any graph G with m vertices, let A represent its adjacency matrix. Then the number of n-step walks from vertex v_x to vertex v_y of length l is the entry in the (x, y) index of A^l, where (x, y) denotes the xth row and yth column [8].*

Proof. We proceed by way of induction.

Base Case. Let $n = 1$. If $n = 1$, then the walk has length 1, so the only valid walks between the two vertices have just one edge between the two vertices. So the number of 1-step walks is equal to the number of edges that are incident to the two vertices. Recall that the number of incident edges can be greater than one due to the idea of "multiple edge" in which adjacent matrices have more than one edge between them. The number of edges that are incident to two vertices is directly written in the (x, y) index of A by the definition of the adjacency matrix. Since $A = A^1$, we conclude that if $n = 1$, the number of n-step walks from vertex v_x to vertex v_y is the (x, y) index of A^n.

Induction Hypothesis. Assume that the proposed theorem is true for when $n = l - 1$, where l is some positive integer. Then the (x, y) index of $A = A^(l - 1)$ is the number of $l - 1$-step walks from v_x to vertex v_y. Let r_{ij} be the value of the (i, j) entry of A^{l-1}, and let p_{ij} be the (i, j) entry of A. To get A^l, we multiply A^{l-1} and A. By matrix multiplication, the (x, y) index of $A^{l-1}A$ would be $r_{x1}p_{1y} + r_{x2}p_{2y} + \cdots + r_{xm}p_{my}$ where A and $A^l - 1$ are m by m matrices. Now, we must show $r_{x1}p_{1y} + r_{x2}p_{2y} + \cdots + r_{xm}p_{my}$ gives the number of l-step walks from v_x to v_y.

Note that any l-step walk from v_x to v_y can be seen as a $(l - 1)$-step walk from v_x to a vertex adjacent to v_y, followed by a 1-step walk from that vertex to v_y. So to count the total number of l-step walks, we sum the number of $l - 1$ step walks to each and every possible vertex adjacent to v_y multiplied by the number of 1-step walks from that vertex to v_y.

Our sum $r_{x1}p_{1y} + r_{x2}p_{2y} + ... + r_{xm}p_{my}$ does exactly what we have observed: r_{x1} gives the number of $(l - 1)$-step walks to v_1 and p_{1y} gives the number of 1-step walks to v_y. So $r_{x1}p_{1y}$ gives the l-step walks between v_x and v_y where v_1 is the second to last vertex. Similarly, $r_{x2}p_{2y}$ gives the number of l-step walks where v_2 is the second to last vertex. This pattern continues all through v_m, thus covering all possible walks since every l-step walk must have some "second to last" vertex. So $A^{l-1}A = A^l$ has the described property that the number of n-step walks of length l is the (x, y) index of A^l. ■

3.2 Connecting Eigenvalues and General n-step Walks

After seeing the capabilities of the adjacency matrix, it's natural to wonder what other aspects of linear algebra play a role in studying graphs. To answer this question, we turn to eigenvalues.

When analyzing any graph G, we define the eigenvalues of G as the eigenvalues of the adjacency matrix of G. For the remainder of this section, we will examine what extra information eigenvalues bring into the picture when evaluating n-step walks.

As stated in Section 2.2, the definition of the adjacency matrix implies it must be *symmetric* across the main diagonal because the number of paths between the (i, j) vertices must be the same number of paths that exist between the (j, i) vertices. This specific classification lends itself to the use of the **Spectral Theorem**, which states that an n by n symmetric matrix must be orthogonally diagonalizable [6]. We now apply this conclusion to the adjacency matrix and derive an explicit formula for the number of n-step walks between two vertices, which is more flexible than the method of always calculating A^l from Theorem 3.1.

Theorem 3.2. *Let A be the adjacency matrix for a graph G with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ (with the same eigenvalue listed multiple times to match its multiplicity). Let $p_{i,j}$ be the entries of P in the orthogonal diagonalization $P^{-1}AP = D$, where D is the diagonalization matrix for A. Then the number of n-step walks between vertices v_x and v_y is given by $c_1\lambda_1^n + \cdots + c_m\lambda_m^n$ where $c_k = p_{x,k}p_{y,k}$ [8].*

Proof. Since the adjacency matrix A is symmetric, we use the Spectral Theorem to conclude that A is orthogonally diagonalizable. Recall that the definition of orthogonally diagonalizable implies that there must exist orthogonal matrices P and P^{-1} such that $P^{-1}AP = D$, where D is a diagonal matrix [6]. Raising D to the nth power, we get $D^n = (P^{-1}AP)^n = (P^{-1}AP)(P^{-1}AP)\ldots(P^{-1}AP) = P^{-1}A(PP^{-1})A\ldots(PP^{-1})AP$. Each paired term PP^{-1} cancels out to the identity matrix I, so the product simplifies to $D^n = P^{-1}A^nP$. Rewriting the equation to isolate A^n, we get $A^n = PD^nP^{-1}$. Using Theorem 3.1, we can say that the number of n-step walks is then the entry in the (x, y) entry of A^n. Since $A^n = PD^nP^{-1}$, we can compute the value of the entry of the (x, y) entry to get the number of n-step walks.

Since D is a diagonalization of A, the entries along the diagonal of D are the eigenvalues of A. So the diagonal of D^n will be $\lambda_1^n, ..., \lambda_m^n$. Also note that since P is an orthogonal matrix, $P^{-1} = P^T$. Using the algebraic formula for matrix multiplication, the value of the (x, y) entry of PD^nP^{-1} is $p_{x1}\lambda_1^n p_{y1} + ... + p_{xm}\lambda_m^n p_{ym}$, which is in the form $c_1\lambda_1^n + \cdots + c_m\lambda_m^n$ where $c_k = p_{x,k}p_{y,k}$. ■

3.3 Connecting Eigenvalues and Closed n-step Walks

In retrospect, the previous method was a bit tedious since we need to compute all eigenvalues as well as the diagonalization matrix. However, there are some cases where the calculation is simpler, specifically when dealing with a n-step walk that is "closed."

Definition 3.2 (Closed)**.** A n-step walk is considered to be **closed** if it *begins and ends at the same vertex.*

We again use eigenvalues to compute the number of closed n-step walks of length l starting and ending from any vertex v_i in the graph, but there is significantly less work involved. To prove the next theorem, we employ the concept of **trace**, which is the *sum of the diagonal entries* of a matrix. We also use the fact that trace is the sum of the eigenvalues of a matrix [1].

Theorem 3.3. *Let A be the adjacency matrix for a graph G with eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_m$ (with the same eigenvalue listed multiple times to match its multiplicity). The number of closed n-step walks on G is given by $\lambda_1^n + \cdots + \lambda_m^n$ [8].*

Proof. Recall that closed walks are walks that begin and end at the same vertex. This can be represented by the diagonal entries of the adjacency matrix, which show the number of walks from a vertex to itself. So the total number of closed n-step walks in a graph G is essentially the sum of the diagonal entries of A^n, where A is the adjacency matrix. By definition, the sum of the diagonal entries of A^n is the trace of the matrix, and trace equals the sum of its eigenvalues [1]. From Theorem 3.2, A^n can be diagonalized into a matrix with $\lambda_1^n, \ldots, \lambda_m^n$ along its diagonal, so the eigenvalues of A^n are $\lambda_1^n, \ldots, \lambda_m^n$. Hence the total number of closed walks is $\lambda_1^n + \ldots + \lambda_m^n$. ■

3.4 Connecting Eigenvalues, n-step Walks, and Complete Graphs

We continue our investigation of how eigenvalues relate to n-step walks here by examining their behavior in complete graph K_p. Recall that these graphs have one edge between any two vertices. We can use the number of vertices – a defining characteristic of a complete graph – to prove the following theorem, which tells us how to calculate the eigenvalues of a complete graph.

Theorem 3.4. *The eigenvalues of a complete graph K_p are $p-1$ with multiplicity 1 and -1 with multiplicity $p-1$ [8].*

Proof. Note that the adjacency matrix A of a complete graph K_p will be a p by p matrix with every entry equal to 1 except on the main diagonal, where all diagonal entries are equal to 0. We can rewrite the adjacency matrix as $A = J - I$, where J is a matrix with all entries of 1, and the identity matrix I is subtracted from J. We know the reduced row echelon form of J has only one pivot column since all rows are identical to each other and are not equal to 0, so J a rank of 1. Since J has rank 1, it can only have one nonzero eigenvalue. We briefly depart to prove this as a lemma.

Lemma 3.5. A matrix with rank 1 has only one nonzero eigenvalue.

Proof. By way of contradiction, assume that a matrix J of rank 1 has more than one nonzero eigenvalue. Let λ_1 and λ_2 be two distinct eigenvalues both nonzero. Let v_1 and v_2 be their respective eigenvectors, making them linearly independent. So Jv_1 and Jv_2 must also be linearly independent. This is impossible since the range of J is one-dimensional. Similarly, this lone nonzero eigenvalue has multiplicity 1, otherwise by the same logic, we would get a list of two linearly independent vectors in range(J). So J has one nonzero eigenvalue and $p-1$ eigenvalues of 0. Since the diagonal is composed of p ones, the trace of J is p, implying that the sum of the eigenvalues is p. Thus J has an eigenvalue of p of multiplicity 1 and another eigenvalue 0 of multiplicity $p-1$. ■

Proceeding with $A = J - I$, we can use the eigenvalues of J to solve for the eigenvalues of A. For any eigenvector e for J, $Ae = (J - I)e = Je - e = \lambda e - e = (\lambda - 1)e$. So every eigenvector of

J is an eigenvector of A, but with an eigenvalue shifted down by 1. Thus the corresponding eigenspaces will be identical, meaning that the multiplicities will be the same. So the eigenvalues of A will be $p-1$ with multiplicity of 1 and $0-1=-1$ with multiplicity of $p-1$. Since A is the adjacency matrix of K_p, this proves the desired statement. ∎

Now that we have a formula to compute the eigenvalues of K_p, we can use it to derive a new formula to calculate the number closed n-step walks on a complete graph.

Theorem 3.6. *For a complete graph K_p, the number of closed n-step walks from a vertex v_x to itself is given by $\frac{(p-1)^n+(p-1)(-1)^n}{p}$ [8].*

Proof. By Theorem 3.4, the eigenvalues of K_p are $p-1$ with multiplicity 1 and -1 with multiplicity $p-1$. So by Theorem 3.3, the total number of closed n-step walks is given by $(p-1)n+(-1)n+...+(-1)n$. By the symmetry of a complete graph, the number of closed walks must be equal for each and every vertex. To find the number of closed n-step walks for a particular vertex, divide the total number by the total number of vertices p, so we arrive at the expression $\frac{(p-1)^n+(p-1)(-1)^n}{p}$. ∎

4 Analysis of Some Examples

Happily equipped with a strong background of linear algebra and graph theory, we now employ our theorems to solve some concrete example problems.

Example 1. Analyze the following simple graph by computing (1) its eigenvalues and (2) the number of closed n-step walks starting and ending at vertex 1.

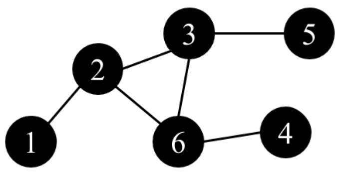


Figure 4.1: Figure by authors.

Solution. (1) First, we need to translate this graph into its adjacency matrix. There are six vertices, so our adjacency matrix will be a 6 by 6 matrix. Given that the graph is simple, we can assume that none of its edges are multiple edges. Thus if an edge exists between vertices i and j, then we mark a 1 for the corresponding index (i, j) of A. Doing so for all vertex combinations, we get:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Now we can continue on our journey to find the eigenvalues of A. We begin by finding the determinant of $A - I\lambda$.

$$det(A - I\lambda) = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 \\ 1 & -\lambda & 1 & 0 & 0 & 1 \\ 0 & 1 & -\lambda & 0 & 1 & 1 \\ 0 & 0 & 0 & -\lambda & 0 & 1 \\ 0 & 0 & 1 & 0 & -\lambda & 0 \\ 0 & 1 & 1 & 1 & 0 & -\lambda \end{vmatrix} = \lambda^6 - 6\lambda^4 - 2\lambda^3 + 6\lambda^2 - 1$$

Factoring the polynomial, we get $(\lambda + \frac{\sqrt{5}+1}{2})^2(\lambda - \frac{\sqrt{5}-1}{2})^2(\lambda + (\sqrt{2}-1))(\lambda - (\sqrt{2}+1))$. So the eigenvalues are $\frac{-\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, -\sqrt{2}+1, \sqrt{2}+1$.

(2) After successfully collecting these eigenvalues, we now seek out the number of closed n-step walks using the explicit rule from Theorem 3.3. Since two of the eigenvalues have a multiplicity of 2, the full list that we input into the formula is $\frac{-\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, -\sqrt{2}+1, \sqrt{2}+1$. We arrive at a general formula for the total number of closed n-step walks:

$$(\frac{-\sqrt{5}-1}{2})^n + (\frac{-\sqrt{5}-1}{2})^n + (\frac{\sqrt{5}-1}{2})^n + (\frac{\sqrt{5}-1}{2})^n + (-\sqrt{2}+1)^n + (\sqrt{2}+1)^n$$

As an extension, say we want to find the total number of closed 1-step walks. It should be clear immediately that since this graph has no loops, there is no way to traverse one edge and get back to the same vertex, so the answer should be 0. Let us verify with Theorem 3.3.

$$(\frac{-\sqrt{5}-1}{2})^1 + (\frac{-\sqrt{5}-1}{2})^1 + (\frac{\sqrt{5}-1}{2})^1 + (\frac{\sqrt{5}-1}{2})^1 + (-\sqrt{2}+1)^1 + (\sqrt{2}+1)^1 =$$
$$2(\frac{-\sqrt{5}-1}{2} + \frac{\sqrt{5}-1}{2}) + (-\sqrt{2}+1+\sqrt{2}+1) = 2(-1) + 2 = 0$$

What about the number of closed 2-step walks? Since 2 is a relatively low number, so we can get to this value visually. From the graph, we can tell that 2-step walk that returns to the vertex it started from involves moving to an adjacent vertex for the first step, and directly coming back for the second step. So the number of closed 2-step walks starting at a single vertex is equal the number of vertices that are adjacent to it. Vertices 1, 4, and 5 each only one adjacent vertex, and 2, 4, and 6 have three adjacent vertexes. So the total number of closed n-step walks is $1+1+1+3+3+3 = 12$. Again, let us verify mathematically with Theorem 3.3.

$$(\frac{-\sqrt{5}-1}{2})^2 + (\frac{-\sqrt{5}-1}{2})^2 + (\frac{\sqrt{5}-1}{2})^2 + (\frac{\sqrt{5}-1}{2})^2 + (-\sqrt{2}+1)^2 + (\sqrt{2}+1)^2 = 12$$

Finally, to get a sense of how quickly the number of n-step walks can escalate, let us compute when $n = 10$.

$$(\frac{-\sqrt{5}-1}{2})^{10} + (\frac{-\sqrt{5}-1}{2})^{10} + (\frac{\sqrt{5}-1}{2})^{10} + (\frac{\sqrt{5}-1}{2})^{10} + (-\sqrt{2}+1)^{10} + (\sqrt{2}+1)^{10} = 6972$$

Example 2. Draw out the complete graph K_{10}. Then, compute (1) its eigenvalues and (2) the number of closed n-step from vertex 1 to itself for when $n = 8$.

Solution. To draw a complete graph with 10 vertices, we plot 10 points and connect every two distinct vertices with an edge.

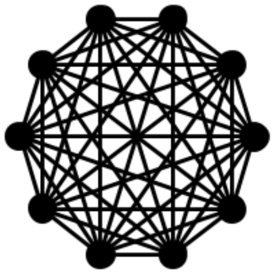


Figure 4.2: An example of a possible pictorial representation for K_{10}. Note that the vertices do not have to be arranged symmetrically as shown. Figure by authors.

(1) By Theorem 3.4, the eigenvalues of K_{10} are 9 with multiplicity 1 and -1 with multiplicity $10-1=9$.

(2) To get the number of closed n-step walks from any vertex to itself, we can use the formula we proved in Theorem 3.5, and get $\frac{(10-1)^n+(10-1)(-1)^n}{10} = \frac{1}{10}(9^n+9(-1)^n)$. Notice that since there are no loops, there should not be any closed 1-step walks. To test this observation, we substitute 1 in for n in the formula, and we indeed get 0. If we try $n=8$, we get $\frac{1}{10}(9^8+9(-1)^8) = 4304673$ n-step walks.

5 Conclusion

Our examination of graph theory only begins to scratch the surface of this fully-fledged branch of mathematics. However, the theorems and ideas we have illustrated with the adjacency matrix and its eigenvalues are inherently fundamental, and may serve as a starting point for those looking to further explore the graph theory. We hope to have provided a new way of viewing graphs for our Math 22A readers, and to have demonstrated the beauty in the relationships between cumulative linear algebra concepts, such as the Spectral Theorem, to applications in graph theory.

To further continue the study of graph theory, there are many paths to take (pun intended *wink*). One possible "next step" is to prove formulas to analyze **random walks**, which have a succession of *random steps* through edges and vertices in a graph. Here one would see the intersection of linear algebra and probability, tying together two profound branches of mathematics to create something new [8].

References

[1] Sheldon Jay Axler. *Linear Algebra Done Right*. Undergraduate Texts in Mathematics. New York: Springer, 1997. ISBN: 0387982582. URL: http://linear.axler.net/.

[2] Kimberly Jordan Burch. *An Introduction to Chemical Graph Theory*. 2008. URL: https://www.maa.org/external_archive/joma/Volume8/Burch/Research.html.

[3] Stephan C. Carlson. *Königsberg bridge problem*. July 2010. URL: https://www.britannica.com/science/Konigsberg-bridge-problem.

[4] John M. Harris, Jeffry L. Hirst, and Michael J. Mossinghoff. *Combinatorics and graph theory*. Springer, 2010.

[5] Pallavi Jayawant and Kerry Glavin. "Minimum spanning trees". In: *Involve* 2.4 (2009), pp. 439–450. DOI: 10.2140/involve.2009.2.439. URL: https://doi.org/10.2140/involve.2009.2.439.

[6] David C. Lay. *Linear Algebra and Its Applications*. Reading, MA: Addison-Wesley, 2003.

[7] Oliver Mason and Mark Verwoerd. "Verwoerd M: Graph theory and networks in biology". In: *IET Systems Biology* (), pp. 1–89.

[8] Richard P. Stanley. *Algebraic combinatorics: walks, trees, tableaux, and more*. Springer., 2018.